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Weak* exactness for dual operator spaces

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Abstract

We introduce a new tensor product $\overset{\sigma,\vee}{\otimes}$: and study the weak* condition C′, which is also called weak* exactness, for dual operator spaces. Our definition of weak* condition C′ is equivalent to Kirchberg's notion of weak exactness in the case of von Neumann algebras. We also study the connection of weak* exact W*-TROs with their linking von Neumann algebras and study the structure of exact (respectively, nuclear) W*-TROs.

Keywords: Dual operator spaces; W*-TROs; von Neumann algebras; Weak* exactness

1. Introduction

In 1980, Archbold and Batty discovered an unexpected complication that can occur for non-nuclear C*-algebras. They introduced two notions, condition C and condition C′, for C*-algebras in [1]. Later on Kirchberg introduced exactness for C*-algebras in [13] and proved in [15] that a C*-algebra is exact if and only if it satisfies condition C′. Kirchberg also showed that condition C′ implies condition C″ introduced by Effros and Haagerup [3] and thus exactness is also equivalent to condition C. The operator space version of this theory has been considered in great detail by Pisier [17] and by Effros, Ozawa and Ruan [7]. The theory has played a significant role in recent

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development of C*-algebras and operator spaces. A convenient reference for C*-algebra results can be found in Wassermann [22] and references for corresponding operator space results can be found in Effros and Ruan [5] and Pisier [18].

However, exactness does not play much role in von Neumann algebra theory since a von Neumann algebra is exact if and only if it is subhomogeneous (see Kirchberg [14, p. 949]). It is natural to explore a suitable analogue of exactness for von Neumann algebras. This motivated Kirchberg to introduce the notion of weak exactness for von Neumann algebras in [14]. Recently, Ozawa investigated this property again and proved a very nice local characterization of weak exactness for von Neumann algebras in [16]. As a corollary, Ozawa showed that if G is a discrete group, then the reduced group C*-algebra $C_{\mathrm{r}}^*(G)$ is exact if and only if the group von Neumann algebra $VN(G)$ is weakly exact. Therefore, we can obtain a non-weakly exact group von Neumann algebra by Gromov's example of non-exact groups.

The goal of this paper is to study weak* exactness for general dual operator spaces. We also study properties associated with weak* exact W*-TROs. The paper is organized as follows. We first introduce a new right augmented tensor product $V \overset{\sigma,\vee}{\otimes}: W^{**}$ for dual operator spaces V and operator spaces W in Section 2. We study some properties of this tensor product and show that for certain class of dual operator spaces V, this new tensor product is injective with respect to arbitrary operator spaces W. However, it is not known whether it is injective for general dual operator spaces. We introduce a corresponding weak* condition C$'$ for dual operator spaces in Section 3. Motivated by Ozawa's results [16, Theorem 2 and Corollary 5], we prove a local characterization of weak* condition C$'$ for dual operator spaces in Theorem 3.3. According to this local characterization result, we say that a dual operator space is weak* exact if it satisfies weak* condition C$'$. As a consequence of this result, it is easy to see that for von Neumann algebras, our definition of weak* condition C$'$ (i.e. weak* exactness) is equivalent to Kirchberg's definition of weak exactness (see definition given in [14,16]).

In [12], Kaur and the second author proved that a TRO (i.e. a ternary ring of operators) V is exact if and only if its linking C*-algebra $A(V)$ is exact. We consider the weak* analogue of this result in Section 4. We are able to show in Theorem 4.1 that a W*-TRO V is weak* exact if and only if its linking von Neumann algebra $R(V)$ is weak* exact. Finally, we study the structure of exact (respectively, nuclear) W*-TROs in Section 5. It is known from Kirchberg [14] (respectively, from Wassermann [21]) that a von Neumann algebra M is exact (respectively, nuclear) if and only if it is subhomogeneous, i.e. we can write

$$M = \prod_{k=1}^{m} \big(M_{n_k} \mathbin{\overline{\otimes}} L_\infty(X_k, \mu_k)\big)$$

with m and n_k $(k = 1, \ldots, m)$ being positive integers. Our main result (Theorem 5.3) in Section 5 shows that a W*-TRO V with a separable predual is exact (respectively, nuclear) if and only if V can be split into the decomposition

$$V = V_{\mathrm{r}} \oplus_\infty V_{\mathrm{c}} \tag{1.1}$$

of a row subhomogeneous W*-TRO

$$V_{\mathrm{r}} = \prod_{k=1}^{\infty} \big(M_{n_k,m_k} \mathbin{\overline{\otimes}} L_\infty(X_k, \mu_k)\big)$$

with $\sup_k\{n_k\} < \infty$ and $m_k \in \mathbf{N} \cup \{\infty\}$, and a column subhomogeneous W*-TRO

$$V_c = \prod_{k=1}^{\infty} \big(M_{n_k,m_k} \overline{\otimes} L_\infty(X_k, \mu_k)\big)$$

with $\sup_k\{m_k\} < \infty$ and $n_k \in \mathbf{N} \cup \{\infty\}$. As a consequence of Theorem 5.3, we can easily show that a dual operator space V is nuclear if and only if V can be decomposed into a direct sum as in (1.1).

Some preliminaries can be found in each section. More details on operator spaces can be found in [5,18].

2. A new right augmented tensor product $\overset{\sigma,\vee}{\otimes}$:

Let us begin this section by recalling the right augmented injective tensor product for operator spaces (see [5, Chapter 14]). Given any linear functional $F : V \otimes W \to \mathbf{C}$ which is bounded in each component (i.e. F is a bounded bilinear functional on $V \times W$), there exists a unique *right weak* continuous* (i.e. weak* continuous in the 2nd component) extension $F_r : V \otimes W^{**} \to \mathbf{C}$ given by

$$\langle F_r, v \otimes w^{**}\rangle = \big\langle w^{**}, F(v \otimes \cdot)\big\rangle \tag{2.1}$$

for all $v \in V$ and $w^{**} \in W^{**}$. If we let $V \check{\otimes} W$ denote the *operator space injective tensor product* of V and W, then we can obtain a canonical inclusion

$$\tau_r : v \otimes w^{**} \in V \otimes W^{**} \to \tau_r(v \otimes w^{**}) \in (V \check{\otimes} W)^{**},$$

which is given by

$$\big\langle \tau_r(v \otimes w^{**}), F\big\rangle = \big\langle w^{**}, F(v \otimes \cdot)\big\rangle = \big\langle F_r, v \otimes w^{**}\big\rangle \tag{2.2}$$

for all $F \in (V \check{\otimes} W)^*$. This inclusion induces an injective operator space tensor product $\check{\otimes}$: on $V \otimes W^{**}$. The completion $V \check{\otimes}: W^{**}$ is called the *right augmented injective tensor product* of V and W^{**}.

Now let us assume that V is a dual operator space with a predual V_* and assume that W is an arbitrary operator space. We let

$$B^\sigma(V \check{\otimes} W, \mathbf{C}) = \big\{F \in (V \check{\otimes} W)^* \;\big|\; F \text{ is left weak* continuous}\big\}, \tag{2.3}$$

where the left weak* continuity of F means that F is weak* continuous in the 1st component. There is a natural operator space matrix norm on $B^\sigma(V \check{\otimes} W, \mathbf{C})$ given by identifying $M_n(B^\sigma(V \check{\otimes} W, \mathbf{C}))$ as a subspace of $M_n((V \check{\otimes} W)^*) = CB(V \check{\otimes} W, M_n)$. We note that each $F \in B^\sigma(V \check{\otimes} W, \mathbf{C})$ is uniquely associated with a completely bounded map $\varphi : W \to V_*$, which is determined by

$$\big\langle \varphi(w), v\big\rangle = F(v \otimes w).$$

Actually, we can prove in Theorem 2.1 that φ is a completely integral map with completely integral norm $\iota(\varphi) = \|F\|$. Let us recall, by an equivalent definition, that a completely bounded map $\varphi : W \to V_*$ is said to be *completely integral* with *completely integral norm* $\iota(\varphi) \leqslant 1$ if there exists a net of contractive elements $F_\alpha \in V_* \hat{\otimes} W^*$, where $V_* \hat{\otimes} W^*$ is the *operator space projective tensor product* of V_* and W^*, such that

$$F_\alpha(v \otimes w) \to \langle v, \varphi(w) \rangle \tag{2.4}$$

for all $v \in V$ and $w \in W$ (see detail in [4] or in [5, Lemma 12.3.1]). We let $\mathcal{I}(W, V_*)$ denote the space of all completely integral maps from W into V_*.

Theorem 2.1. *Given a dual operator space V with a predual V_* and an operator space W, we have the completely isometric isomorphism*

$$B^\sigma(V \check{\otimes} W, \mathbf{C}) = \mathcal{I}(W, V_*).$$

Proof. Let us first recall from [5, Lemma 12.3.3] that there exists a completely isometric injection

$$S_0 : \varphi \in \mathcal{I}(W, V_*) \to S_0(\varphi) \in (V \check{\otimes} W)^*$$

given by

$$S_0(\varphi)(v \otimes w) = \langle v, \varphi(w) \rangle$$

for all $v \in V$ and $w \in W$. Since $\varphi(w) \in V_*$ for every $w \in W$, $S_0(\varphi)$ is left weak* continuous and thus is contained in $B^\sigma(V \check{\otimes} W, \mathbf{C})$. Therefore, S_0 is a complete isometry from $\mathcal{I}(W, V_*)$ into $B^\sigma(V \check{\otimes} W, \mathbf{C})$. We only need to prove that S_0 is onto.

Let us assume that $F \in B^\sigma(V \check{\otimes} W, \mathbf{C}) \subseteq (V \check{\otimes} W)^*$ with $\|F\| \leqslant 1$. Then there exists a completely bounded map $\varphi \in CB(W, V^*) \cong (V \hat{\otimes} W)^*$ such that

$$\langle v, \varphi(w) \rangle = F(v \otimes w)$$

for all $v \in V$ and $w \in W$. Since $F \in B^\sigma(V \check{\otimes} W, \mathbf{C})$ is left weak* continuous, we get $\varphi \in CB(W, V_*)$. In the following, we want to show that $\varphi \in \mathcal{I}(W, V_*)$. Since we have the completely isometric inclusions

$$V \check{\otimes} W \hookrightarrow CB(V_*, W) \hookrightarrow CB(V_*, W^{**}) = (V_* \hat{\otimes} W^*)^*,$$

F has a contractive extension $\widetilde{F} \in (V_* \hat{\otimes} W^*)^{**}$. By bipolar theorem, we can obtain a net of contractive elements $\{F_\alpha\} \subseteq V_* \hat{\otimes} W^*$ such that $F_\alpha \to \widetilde{F}$ with respect to the $\sigma((V_* \hat{\otimes} W^*)^{**}, (V_* \hat{\otimes} W^*)^*)$ topology. Now for each $v \in V$ and $w \in W$, we have

$$F_\alpha(v \otimes w) \to \widetilde{F}(v \otimes w) = F(v \otimes w) = \langle v, \varphi(w) \rangle.$$

According to the definition given in (2.4), this shows that $\varphi \in \mathcal{I}(W, V_*)$ with $\iota(\varphi) \leqslant 1$. Therefore, the map S_0 is onto. $\square$

For each $F \in B^\sigma(V \check{\otimes} W, \mathbf{C})$, it is clear that $F_r : V \otimes W^{**} \to \mathbf{C}$ is right weak* continuous. The following result shows that F_r is also left weak* continuous.

Proposition 2.2. *Given any $F \in B^\sigma(V \check{\otimes} W, \mathbf{C})$, its right weak* continuous extension $F_r : V \otimes W^{**} \to \mathbf{C}$ is bi-weak* continuous on $V \otimes W^{**}$.*

Proof. Let $F \in B^\sigma(V \check{\otimes} W, \mathbf{C})$ and let φ be the corresponding completely integral map contained in $\mathcal{I}(W, V_*)$. It is known from [5, Corollaries 13.2.2 and 13.3.4] that φ is completely 1-summing and thus can be factored through a column Hilbert space $\mathcal{H}_c$. Therefore, φ is weakly compact and thus φ^{**}, which corresponds to F_r, maps W^{**} into V_*. More precisely, let us assume that $\varphi = \beta \circ \alpha$ with

$$\alpha : W \to \mathcal{H}_c \quad \text{and} \quad \beta : \mathcal{H}_c \to V_*.$$

Then for any $v \in V$ and $w^{**} \in W^{**}$, we have

$$\begin{aligned} F_r(v \otimes w^{**}) &= \langle w^{**}, F(v \otimes \cdot)\rangle = \langle w^{**}, \varphi^*(v)\rangle \\ &= \langle w^{**}, \alpha^* \circ \beta^*(v)\rangle = \langle \beta(\alpha^{**}(w^{**})), v\rangle, \end{aligned}$$

where we used the fact that $\alpha^{**}(W^{**}) \subseteq \mathcal{H}_c^{**} = \mathcal{H}_c$ in the last equality. This shows that $F_r(\cdot \otimes w^{**}) = \beta \circ \alpha^{**}(w^{**}) \in V_*$ for all $w^{**} \in W^{**}$. Therefore, F_r is left weak* continuous. $\square$

As we discussed in (2.2), we can obtain a canonical inclusion

$$\tau_r^\sigma : V \otimes W^{**} \to B^\sigma(V \check{\otimes} W, \mathbf{C})^*$$

and thus obtain a new tensor norm on $V \otimes W^{**}$. We let $V \overset{\sigma,\vee}{\otimes:} W^{**}$ denote its completion. This definition behaves well with respect to matrices. Indeed, if we let $T_n = M_n^*$ denote the operator dual of M_n, we can conclude from [5, Corollary 12.3.5] and Theorem 2.1 that

$$\begin{aligned} M_n(B^\sigma(V \check{\otimes} W, \mathbf{C})^*) &= (T_n \hat{\otimes} B^\sigma(V \check{\otimes} W, \mathbf{C}))^* = (T_n \hat{\otimes} \mathcal{I}(W, V_*))^* \\ &= (\mathcal{I}(M_n(W), V_*))^* = B^\sigma(V \check{\otimes} M_n(W), \mathbf{C})^*. \end{aligned}$$

This gives us the isometry

$$M_n(V \overset{\sigma,\vee}{\otimes:} W^{**}) = V \overset{\sigma,\vee}{\otimes:} M_n(W)^{**}. \tag{2.5}$$

Since $T_n \hat{\otimes} \mathcal{I}(W, V_*) = \mathcal{I}(W, T_n \hat{\otimes} V_*) = \mathcal{I}(W, M_n(V)_*)$, we can similarly obtain

$$M_n(V \overset{\sigma,\vee}{\otimes:} W^{**}) = M_n(V) \overset{\sigma,\vee}{\otimes:} W^{**}. \tag{2.6}$$

Therefore, we can obtain a natural operator space structure on $V \overset{\sigma,\vee}{\otimes:} W^{**}$.

If we are given a dual operator space V and operator spaces $W \subseteq \widetilde{W}$, it is easy to see that the (right) restriction map

$$R : \widetilde{F} \in B^\sigma(V \check{\otimes} \widetilde{W}, \mathbf{C}) \to F = \widetilde{F}|_{V \check{\otimes} W} \in B^\sigma(V \check{\otimes} W, \mathbf{C})$$

is a complete contraction and the restriction of its adjoint map

$$R^* : B^\sigma(V \check{\otimes} W, \mathbf{C})^* \to B^\sigma(V \check{\otimes} \widetilde{W}, \mathbf{C})^*$$

to $V \overset{\sigma,\vee}{\otimes}: W^{**}$ is just the canonical inclusion

$$id_V \otimes \iota^{**} : V \overset{\sigma,\vee}{\otimes}: W^{**} \to V \overset{\sigma,\vee}{\otimes}: \widetilde{W}^{**}. \tag{2.7}$$

In the following, we show that for certain classes of dual operator spaces V, this canonical inclusion $id_V \otimes \iota^{**}$ is a completely isometric injection. However, we cannot prove such a result for general dual operator spaces and thus we cannot say that $\overset{\sigma,\vee}{\otimes}:$ is an injective tensor product.

Proposition 2.3. *Let M be a von Neumann algebra. Then for any operator spaces $W \subseteq \widetilde{W}$, the restriction map*

$$R : \widetilde{F} \in B^\sigma(M \check{\otimes} \widetilde{W}, \mathbf{C}) \to F = \widetilde{F}|_{M \check{\otimes} W} \in B^\sigma(M \check{\otimes} W, \mathbf{C})$$

is a complete quotient. Therefore, the canonical inclusion

$$id_M \otimes \iota^{**} : \ M \overset{\sigma,\vee}{\otimes}: W^{**} \to M \overset{\sigma,\vee}{\otimes}: \widetilde{W}^{**}$$

is a completely isometric injection.

Proof. Let us assume that $\widetilde{W} \subseteq B$ is an operator subspace of a unital C*-algebra B. Given any contraction $F \in B^\sigma(M \check{\otimes} W, \mathbf{C}) \subseteq (M \check{\otimes} W)^*$, we let $G \in (M \check{\otimes} B)^*$ be a contractive extension of F. Then there exist a Hilbert space $\mathcal{H}$, a $*$-representation $\pi : M \check{\otimes} B \to \mathcal{B}(\mathcal{H})$, and contractive vectors $\xi, \eta \in \mathcal{H}$ such that

$$G(x \otimes b) = \eta^* \pi(x \otimes b)\xi, \quad \forall x \in M, b \in B.$$

Now we can write

$$\pi = \pi_M \cdot \pi_B,$$

where $\pi_M(x) = \pi(x \otimes 1)$ and $\pi_B(b) = \pi(1 \otimes b)$. Furthermore, we can decompose

$$\pi_M = \pi_M^{\mathrm{n}} \oplus \pi_M^{\mathrm{s}},$$

where π_M^{n} is the normal part of π_M and π_M^{s} is the singular part of π_M. Since

$$\left(\pi_M^{\mathrm{s}}\right)^* : \mathcal{B}(\mathcal{H})_* \to M_*^\perp \quad \text{and} \quad \left(\pi_M^{\mathrm{n}}\right)^* : \mathcal{B}(\mathcal{H})_* \to M_*,$$

it follows from the left weak* continuity of F that for each $b \in W$,

$$F(\cdot \otimes b) = \eta^* \pi_M^{\mathrm{n}}(\cdot)\pi_B(b)\xi + \eta^* \pi_M^{\mathrm{s}}(\cdot)\pi_B(b)\xi = \eta^* \pi_M^{\mathrm{n}}(\cdot)\pi_B(b)\xi.$$

Since there exists a central projection $z \in \pi_M(M)''$ such that $\pi_M^{\mathrm{n}}(x) = z\pi_M(x)$ and since π_B has a commuting range with π_M, we can write

$$\pi_M^{\mathrm{n}} \cdot \pi_B(x \otimes b) = \pi_M^{\mathrm{n}}(x)\pi_B(b) = z\pi_M(x)\pi_B(b) = z\pi(x \otimes b)$$

for all $x \in M$ and $b \in B$. This shows that $\pi_M^{\mathrm{n}} \cdot \pi_B$ is again a $*$-representation from $M \check{\otimes} B$ into $\mathcal{B}(\mathcal{H})$. Therefore, $\widetilde{F} : M \check{\otimes} \widetilde{W} \to \mathbf{C}$ defined by

$$\widetilde{F}(x \otimes b) = \eta^* \pi_M^{\mathrm{n}}(x)\pi_B(b)\xi$$

is a contractive linear extension of F. It is clear that $\widetilde{F}$ is left weak* continuous and thus is contained in $B^\sigma(M \check{\otimes} \widetilde{W}, \mathbf{C})$. This shows that the restriction map R is a quotient from $B^\sigma(M \check{\otimes} \widetilde{W}, \mathbf{C})$ onto $B^\sigma(M \check{\otimes} W, \mathbf{C})$. We can prove that R is a complete quotient by a standard matricial argument.

In this case, the adjoint map

$$R^* : B^\sigma(M \check{\otimes} W, \mathbf{C})^* \to B^\sigma(M \check{\otimes} \widetilde{W}, \mathbf{C})^*$$

is a completely isometric injection. Then the following commutative diagram

$$\begin{tikzcd}
M \otimes W^{**} \arrow[r, hook] \arrow[d] & B^\sigma(M \check{\otimes} W, \mathbf{C})^* \arrow[d] \\
M \otimes \widetilde{W}^{**} \arrow[r, hook] & B^\sigma(M \check{\otimes} \widetilde{W}, \mathbf{C})^*
\end{tikzcd}$$

shows that the canonical inclusion $id_M \otimes \iota^{**} : M \overset{\sigma,\vee}{\otimes} W^{**} \to M \overset{\sigma,\vee}{\otimes} \widetilde{W}^{**}$ is a completely isometric injection. □

Proposition 2.3 shows that if $X = M_*$ is the predual of a von Neumann algebra, then for any operator spaces $W \subseteq \widetilde{W}$, the restriction map

$$R : \varphi \in \mathcal{I}(\widetilde{W}, X) \to \varphi|_W \in \mathcal{I}(W, X)$$

is a complete quotient. It is worthy to note that this result is still true if X is an operator space such that X is completely contractively complemented in its second dual X^{**} and we have the complete isometry $\mathcal{I}(W, X) = \mathcal{I}^{\mathrm{ex}}(W, X)$ for all operator spaces W, where $\mathcal{I}^{\mathrm{ex}}(W, X)$ is the space of all exactly integral maps from W into X. This class of operator spaces includes injective C*-algebras, injective operator spaces (equivalently, injective TROs), von Neumann algebras and W*-TROs.

Let us recall from [5, Chapter 15] that a completely bounded map $\varphi : W \to X$ is said to be *exactly integral* with *exactly integral norm* $\iota^{\mathrm{ex}}(\varphi) \leqslant 1$ if it has a factorization

$$\begin{tikzcd} \mathcal{B}(\mathcal{H}) \arrow[r, "M(\omega)"] & \mathcal{B}(\mathcal{K})^* \arrow[dr, "\tilde{s}"] & \\ W \arrow[u, "r"] \arrow[r, "\varphi"] & X \arrow[r, hook, "\iota_X"] & X^{**}, \end{tikzcd} \tag{2.8}$$

where $r : W \to \mathcal{B}(\mathcal{H})$ and $\tilde{s} : \mathcal{B}(\mathcal{K})^* \to X^{**}$ are complete contractions and ω is a contractive linear functional on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. The map $M(\omega) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})^*$ is defined by $M(\omega)(x)(y) = \omega(x \otimes y)$ for $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$. It is also known (see [5, Corollary 12.3.7]) that a map φ is completely integral with $\iota(\varphi) \leqslant 1$ if and only if φ has a factorization (2.8) with $\tilde{s} : \mathcal{B}(\mathcal{K})^* \to X^{**}$ being weak* continuous, i.e. $\tilde{s} = s^*$ for some $s : X^* \to \mathcal{B}(\mathcal{K})$. Therefore, we always have $\mathcal{I}(W, X) \subseteq \mathcal{I}^{\mathrm{ex}}(W, X)$ and have $\iota^{\mathrm{ex}}(\varphi) \leqslant \iota(\varphi)$ for all $\varphi \in \mathcal{I}(W, X)$. It is a quite surprising result of [6] (see [5, Lemma 15.2.2]) that if X is a C*-algebra, then for any operator space W we have the complete isometry

$$\mathcal{I}(W, X) = \mathcal{I}^{\mathrm{ex}}(W, X).$$

Now we can obtain the following result.

Proposition 2.4. *Let V be a dual operator space such that V_* is completely contractively complemented in V^* and we have the isometry $\mathcal{I}(W, V_*) = \mathcal{I}^{\mathrm{ex}}(W, V_*)$ for all operator spaces W. Then for any operator spaces $W \subseteq \widetilde{W}$, the restriction map*

$$R : \varphi \in \mathcal{I}(\widetilde{W}, V_*) \to \varphi|_W \in \mathcal{I}(W, V_*)$$

is a complete quotient and the canonical inclusion

$$id_V \otimes \iota^{**} : V \overset{\sigma,\vee}{\otimes} W^{**} \to V \overset{\sigma,\vee}{\otimes} \widetilde{W}^{**}$$

is a completely isometric injection.

Proof. Let P be a completely contractive projection from V^* onto V_* such that $P \circ \iota_{V_*} = id_{V_*}$. For any $\varphi \in \mathcal{I}(W, V_*)$ with $\iota(\varphi) \leqslant 1$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{B}(\mathcal{H}) \arrow[r, "M(\omega)"] & \mathcal{B}(\mathcal{K})^* \arrow[dr, "s^*"] & \\ W \arrow[u, "r"] \arrow[r, "\varphi"] & V_* \arrow[r, hook, "\iota_{V_*}"] & V^*, \end{tikzcd}$$

where $r : W \to \mathcal{B}(\mathcal{H})$ and $s : V \to \mathcal{B}(\mathcal{K})$ are complete contractions and ω is a contractive linear functional on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. For any operator space $\widetilde{W}$ containing W as an operator subspace, r has a completely contractive extension $\tilde{r} : \widetilde{W} \to \mathcal{B}(\mathcal{H})$. Then $\tilde{\varphi} = P \circ s^* \circ M(\omega) \circ \tilde{r}$ is an exactly integral map from $\widetilde{W}$ into V_* with $\iota^{\mathrm{ex}}(\tilde{\varphi}) \leqslant 1$. Since we have the isometric isomorphism

$\mathcal{I}(\widetilde{W}, V_*) = \mathcal{I}^{\mathrm{ex}}(\widetilde{W}, V_*)$, the map $\tilde{\varphi}$ is an extension of φ with $\iota(\tilde{\varphi}) = \iota^{\mathrm{ex}}(\tilde{\varphi}) \leqslant 1$. Therefore, the restriction map R is a quotient from $\mathcal{I}(\widetilde{W}, V_*)$ onto $\mathcal{I}(W, V_*)$. A standard matricial argument shows that R is a complete quotient. □

We note that if V_* is a reflexive operator space, then we have

$$\mathcal{I}(W, V_*) = \mathcal{I}(W, V^*) = (V \check{\otimes} W)^* \tag{2.9}$$

for all operator spaces W. Given operator spaces $W \subseteq \widetilde{W}$,

$$id_V \otimes \iota : V \check{\otimes} W \hookrightarrow V \check{\otimes} \widetilde{W}$$

is a completely isometric injection and thus

$$R = (id_V \otimes \iota)^* : \mathcal{I}(\widetilde{W}, V_*) = (V \check{\otimes} \widetilde{W})^* \to \mathcal{I}(W, V_*) = (V \check{\otimes} W)^*$$

is a complete quotient. Therefore,

$$id_V \otimes \iota^{**} : V \overset{\sigma,\vee}{\otimes} W^{**} \to V \overset{\sigma,\vee}{\otimes} \widetilde{W}^{**}$$

is a completely isometric injection. This shows that such a result also holds for reflexive operator spaces. On the other hand, we may have the following result for the inclusions on dual operator space side.

Proposition 2.5. *Let V_0 be a weak* closed subspace of a dual operator space V such that there exists a weak* continuous completely contractive projection $P : V \to V_0$. Then for any operator space W, the canonical inclusion*

$$\iota \otimes id_{W^{**}} : V_0 \overset{\sigma,\vee}{\otimes} W^{**} \to V \overset{\sigma,\vee}{\otimes} W^{**}$$

is a completely isometric injection.

Proof. It is clear that the (left) restriction map

$$L : \widetilde{F} \in B^\sigma(V \check{\otimes} W, \mathbf{C}) \to F = \widetilde{F}|_{V_0 \check{\otimes} W} \in B^\sigma(V_0 \check{\otimes} W, \mathbf{C})$$

is a complete contraction. On the other hand, for every $F \in B^\sigma(V_0 \check{\otimes} W, \mathbf{C})$, it is clear that $\widetilde{F} = F \circ (P \otimes id_W)$ is contained in $B^\sigma(V \check{\otimes} W, \mathbf{C})$ such that $F = \widetilde{F}|_{V_0 \check{\otimes} W}$ and $\|\widetilde{F}\| = \|F\|$. This shows that L is a quotient map from $B^\sigma(V \check{\otimes} W, \mathbf{C})$ onto $B^\sigma(V_0 \check{\otimes} W, \mathbf{C})$. A standard matricial argument shows that L is a complete quotient. Therefore, the restriction of its adjoint map L^* to $V_0 \overset{\sigma,\vee}{\otimes} W^{**}$ induces a completely isometric injection from $V_0 \overset{\sigma,\vee}{\otimes} W^{**}$ into $V \overset{\sigma,\vee}{\otimes} W^{**}$. □

To end this section, we prove the following result related to C*-algebras.

Proposition 2.6. *Suppose that M is a von Neumann algebra and B is a unital C*-algebra. Then $M \overset{\sigma,\vee}{\otimes} B^{**}$ is a unital* C*-*algebra.*

Proof. We first claim that $B^\sigma(M \check{\otimes} B, \mathbf{C})$ is an $M \check{\otimes} B$ invariant closed subspace of $(M \check{\otimes} B)^*$. Then there exists a central projection z in $(M \check{\otimes} B)^{**}$ such that

$$B^\sigma(M \check{\otimes} B, \mathbf{C}) = z \cdot (M \check{\otimes} B)^*$$

(see [20, Theorem III 2.7]) and thus $B^\sigma(M \check{\otimes} B, \mathbf{C})^*$ is completely isometric to $z \cdot (M \check{\otimes} B)^{**}$. Every element $x \otimes y^{**} \in M \overset{\sigma,\vee}{\otimes}: B^{**}$ can be identified with $z \cdot (x \otimes y^{**})$ in $M \check{\otimes}: B^{**} \subseteq (M \check{\otimes} B)^{**}$. Therefore, we can obtain a C*-algebra norm on $M \overset{\sigma,\vee}{\otimes}: B^{**}$ given by

$$\|u\|_{M \overset{\sigma,\vee}{\otimes}: B^{**}} = \|z \cdot u\|_{(M \check{\otimes} B)^{**}}$$

for all $u = \sum_i x_i \otimes y_i^{**} \in M \otimes B^{**}$.

To see the claim, we let $F \in B^\sigma(M \check{\otimes} B, \mathbf{C})$. Then there exist a Hilbert space $\mathcal{H}$, a unital $*$-homomorphism $\pi : M \check{\otimes} B \to \mathcal{B}(\mathcal{H})$ such that π is left weak* continuous (see argument given in Proposition 2.3), and $\xi, \eta \in \mathcal{H}$ such that

$$F(u) = \eta^* \pi(u)\xi.$$

Then $\pi_M = \pi(\cdot \otimes 1)$ and $\pi_B = \pi(1 \otimes \cdot)$ are unital $*$-homomorphisms from M and B into $\mathcal{B}(\mathcal{H})$, respectively. They have commuting ranges and we may assume that π_M is weak* continuous. In this case, we can write $\pi = \pi_M \cdot \pi_B$. Since

$$u \cdot F(x \otimes y) = F\big((x \otimes y)u\big) = \eta^* \pi(x \otimes y)\pi(u)\xi = \eta^* \pi_M(x)\big(\pi_B(y)\pi(u)\xi\big),$$

this shows that $u \cdot F$ is a bounded left weak* continuous functional contained in $B^\sigma(M \check{\otimes} B, \mathbf{C})$. Similarly, we can proved that $F \cdot u \in B^\sigma(M \check{\otimes} B, \mathbf{C})$. Therefore, $B^\sigma(M \check{\otimes} B, \mathbf{C})$ is an $M \check{\otimes} B$ invariant closed subspace of $(M \check{\otimes} B)^*$. □

3. Weak* condition C′ and weak* exactness for dual operator spaces

It is clear that the canonical bilinear map

$$(f, g) \in V_* \times W^* \to f \otimes g \in B^\sigma(V \check{\otimes} W, \mathbf{C}) \subseteq (V \check{\otimes} W)^*$$

extends to a complete contraction from $V_* \hat{\otimes} W^*$ into $B^\sigma(V \check{\otimes} W, \mathbf{C}) \subseteq (V \check{\otimes} W)^*$. Taking the adjoints, we obtain complete contractions

$$(V \check{\otimes} W)^{**} \to B^\sigma(V \check{\otimes} W, \mathbf{C})^* \to (V_* \hat{\otimes} W^*)^*.$$

Since

$$V \check{\otimes} W^{**} \hookrightarrow (V^* \hat{\otimes} W^*)^* = CB(V^*, W^{**})$$

is a completely isometric inclusion, it is easy to see that the identity map on $V \otimes W^{**}$ extends to complete contractions

$$V \check{\otimes}: W^{**} \to V \overset{\sigma,\vee}{\otimes}: W^{**} \to V \check{\otimes} W^{**}. \tag{3.1}$$

An operator space V is said to satisfy *condition* C' if for any operator space W, we have the completely isometric isomorphism

$$V \check{\otimes}\!\!: W^{**} = V \check{\otimes} W^{**}.$$

Motivated by this definition, we say that a dual operator space V satisfies *weak* condition* C' if for any operator space W, we have the completely isometric isomorphism

$$V \overset{\sigma,\vee}{\otimes}\!\!: W^{**} = V \check{\otimes} W^{**}. \tag{3.2}$$

It is an immediate consequence of (3.1) that weak* condition C' is weaker than condition C' on dual operator spaces V. It is also easy to see from (2.9) that a reflexive operator space V satisfies condition C' if and only if it satisfies weak* condition C' since in this case, we always have the complete isometry $V \check{\otimes}\!\!: W^{**} = V \overset{\sigma,\vee}{\otimes}\!\!: W^{**}$ for any operator space W. We can also easily obtain the following result by applying Proposition 2.5.

Proposition 3.1. *Assume that V_0 is a weak* closed subspace in a dual operator space V and assume that there exists a weak* continuous completely contractive projection $P : V \to V_0$. If V satisfies weak* condition C', then so is V_0.*

It has been observed by Han [9] that an operator space V satisfies condition C' if and only if for any operator space W and any contractive linear functional $F \in (V \check{\otimes} W)^*$, its right weak* continuous extension F_{r} is contractive on $V \otimes_\vee W^{**}$, where we let $V \otimes_\vee W^{**}$ denote the algebraic tensor product $V \otimes W^{**}$ equipped with the injective tensor norm $\|\cdot\|_\vee$. In this case, F_{r} extends to a contraction on $V \check{\otimes} W^{**}$. We can obtain the corresponding analogue for weak* condition C'.

Proposition 3.2. *Let V be a dual operator space. Then the following are equivalent*:

(1) *V satisfies weak* condition* C';
(2) *for any operator space W, any Hilbert space $\mathcal{H}$, and any left weak* continuous complete contraction $\varphi : V \check{\otimes} W \to \mathcal{B}(\mathcal{H})$, its right weak* continuous extension φ_{r} is completely contractive on $V \otimes_\vee W^{**}$*;
(3) *for any operator space W and any contractive linear functional $F \in B^\sigma(V \check{\otimes} W, \mathbf{C})$, its right weak* continuous extension F_{r} is contractive on $V \otimes_\vee W^{**}$.*

Proof. (1) $\Rightarrow$ (2). Let W be an arbitrary operator space and $\varphi : V \check{\otimes} W \to \mathcal{B}(\mathcal{H})$ a left weak* continuous complete contraction. Then φ induces a complete contraction

$$\Phi : \mathcal{B}(\mathcal{H})_* \to B^\sigma(V \check{\otimes} W, \mathbf{C})$$

given by $\Phi(\omega) = \omega \circ \varphi$. The adjoint map

$$\Phi^* : B^\sigma(V \check{\otimes} W, \mathbf{C})^* \to \mathcal{B}(\mathcal{H})$$

is a complete contraction from $B^\sigma(V \check{\otimes} W, \mathbf{C})^*$ into $\mathcal{B}(\mathcal{H})$ such that

$$\langle \varphi_{\mathrm{r}}(v \otimes w^{**}), \omega \rangle = \langle v \otimes w^{**}, (\omega \circ \varphi)_{\mathrm{r}} \rangle = \langle \Phi^*(v \otimes w^{**}), \omega \rangle$$

for all $v \in V$, $w^{**} \in W^{**}$ and $\omega \in \mathcal{B}(\mathcal{H})_*$. Then the right weak* continuous extension

$$\varphi_{\mathrm{r}} = \Phi^*|_{V \otimes W^{**}} : V \otimes W^{**} \to \mathcal{B}(\mathcal{H})$$

is a well-defined (bi-weak* continuous) complete contraction with respect to the new right augmented tensor norm. Now if V satisfies weak* condition C′, then we have the complete isometry $V \check{\otimes} W^{**} = V \overset{\sigma,\vee}{\otimes :} W^{**}$ and thus φ_{r} is a complete contraction from $V \otimes_\vee W^{**}$ into $\mathcal{B}(\mathcal{H})$.

(2) ⇒ (3) is obvious.

(3) ⇒ (1). Given any $z \in V \otimes W^{**}$, we always have

$$\|z\|_{V \check{\otimes} W^{**}} \leqslant \|z\|_{V \overset{\sigma,\vee}{\otimes :} W^{**}}.$$

On the other hand, it follows from the hypothesis that

$$\begin{aligned}\|z\|_{V \overset{\sigma,\vee}{\otimes :} W^{**}} &= \sup\{|\langle F, \tau_{\mathrm{r}}^\sigma(z)\rangle|: \forall F \in B^\sigma(V \check{\otimes} W, \mathbf{C})_1\} \\ &= \sup\{|\langle F_{\mathrm{r}}, z\rangle|: \forall F \in B^\sigma(V \check{\otimes} W, \mathbf{C})_1\} \\ &\leqslant \sup\{|\langle \widetilde{F}, z\rangle|: \forall \widetilde{F} \in (V \check{\otimes} W^{**})_1^*\} = \|z\|_{V \check{\otimes} W^{**}}.\end{aligned}$$

Therefore, we have the isometric isomorphism $V \overset{\sigma,\vee}{\otimes :} W^{**} = V \check{\otimes} W^{**}$. Since W is arbitrary, we may use (2.5) to obtain the isometric isomorphisms

$$M_n\big(V \overset{\sigma,\vee}{\otimes :} W^{**}\big) = V \overset{\sigma,\vee}{\otimes :} M_n(W)^{**} = V \check{\otimes} M_n(W)^{**} = M_n(V \check{\otimes} W^{**})$$

for all $n \in \mathbf{N}$. This shows that V satisfies weak* condition C′. □

The following theorem is a natural dual operator space analogue of Ozawa's local characterization results for von Neumann algebras in [16, Corollary 5 and (1) and (2) in Theorem 2]. The proofs of (1) ⇒ (2) ⇒ (3) are similar to the corresponding arguments given in [16]. The proof of (3) ⇒ (1) is new.

Theorem 3.3. *Suppose that V is a dual operator space. Then the following are equivalent*:

(1) *V satisfies weak* condition* C′;
(2) *for any operator space W and any finite rank complete contraction $\varphi : W^* \to V$, there exists a net of weak* continuous finite rank complete contractions $\varphi_\alpha : W^* \to V$ which converges to φ in the point-weak* topology*;
(3) *for any finite-dimensional operator subspace $E \subseteq V$, there exist nets of complete contractions $\varphi_i : E \to M_{n(i)}$ and $\psi_i : \varphi_i(E) \to V$ such that the net $\{\psi_i \circ \varphi_i\}$ converges to id_E in the point-weak* topology.*

Proof. (1) ⇒ (2). Since V satisfies weak* condition C′, we have the completely isometric inclusion

$$V \check{\otimes} W^{**} = V \overset{\sigma,\vee}{\otimes :} W^{**} \hookrightarrow B^\sigma(V \check{\otimes} W, \mathbf{C})^*.$$

Assume that φ is a finite rank complete contraction from W^* into V. Then it follows from the Hahn–Banach theorem that

$$\varphi \in V \check{\otimes} W^{**} = V \overset{\sigma,\vee}{\otimes} W^{**} \hookrightarrow B^{\sigma}(V \check{\otimes} W, \mathbf{C})^*$$

has a norm preserving extension $\tilde{\varphi} : (V \check{\otimes} W)^* \to \mathbf{C}$, i.e., $\tilde{\varphi} \in (V \check{\otimes} W)_1^{**}$. We can choose a net $\{\varphi_\alpha\} \subseteq V \otimes_\vee W \subseteq (V \check{\otimes} W)^{**}$ with $\|\varphi_\alpha\|_\vee \leqslant 1$ and $\varphi_\alpha \to \tilde{\varphi}$ in the $\sigma((V \check{\otimes} W)^{**}, (V \check{\otimes} W)^*)$ topology. Since $\varphi_\alpha \in V \otimes_\vee W$, $\{\varphi_\alpha\}$ is a net of weak* continuous finite rank complete contractions from W^* into V. For any $f \in W^*$, $g \in V_*$, we get $g \otimes f \in V_* \otimes W^* \subseteq B^{\sigma}(V \check{\otimes} W, \mathbf{C}) \subseteq (V \check{\otimes} W)^*$ and thus have

$$\begin{aligned}\langle \varphi_\alpha(f), g \rangle &= \langle \varphi_\alpha, g \otimes f \rangle \to \langle \tilde{\varphi}, g \otimes f \rangle \\ &= \langle \varphi, g \otimes f \rangle = \langle \varphi(f), g \rangle.\end{aligned}$$

This shows that $\varphi_\alpha \to \varphi$ in the point-weak* topology.

(2) $\Rightarrow$ (3). Let $V \subseteq \mathcal{B}(\mathcal{H})$ and $E \subseteq V$ be a finite-dimensional operator space. We let $\mathcal{F}$ denote the collection of all finite-dimensional subspaces of $\mathcal{H}$ and $\mathcal{F}$ have the partial ordering $F \preccurlyeq F'$ if $F \subseteq F'$. The family $\mathcal{U}_0$ of intervals

$$I(F) = \{F' \in \mathcal{F}\colon\ F \preccurlyeq F'\}$$

is a filter on $\mathcal{F}$ and we let $\mathcal{U}$ be an ultrafilter containing $\mathcal{U}_0$. For any $F \in \mathcal{F}$, we let $\Phi_F : E \to \mathcal{B}(F) = M_{n(F)}$ denote the compression corresponding to F, where $n(F) = \dim F$. As the proof given in [17] or in [16], we may select a finite-dimensional subspace $F_0 \in \mathcal{F}$ such that Φ_F is a linear isomorphism from E onto $\Phi_F(E)$ for $F_0 \preccurlyeq F$ in $\mathcal{F}$. We let $\Phi_F^{-1} = 0$ for $F \notin I(F_0)$. Consider the complete isometry

$$\Phi : a \in E \to \big(\Phi_F(a)\big)_{F \in \mathcal{F}} \in \prod_{F \in \mathcal{F}} \Phi_F(E).$$

We note that $\prod_{F \in \mathcal{F}} \Phi_F(E) \subseteq \prod_{F \in \mathcal{F}} M_{n(F)}$ is a weak* closed operator subspace and hence is a dual operator space. The map Φ has a left inverse given by

$$\Psi : (x_F)_{F \in \mathcal{F}} \in \prod_{F \in \mathcal{F}} \Phi_F(E) \to w^*\text{-}\lim_{\mathcal{U}} \Phi_F^{-1}(x_F) \in E \subseteq V.$$

Indeed, Ψ is a well-defined finite rank complete contraction such that $\Psi \circ \Phi = id_E$. From (2), we can obtain a net of finite rank weak* continuous complete contractions $\psi_\alpha : \prod_{F \in \mathcal{F}} \Phi_F(E) \to V$ such that $\lim_\alpha \psi_\alpha = \Psi$ in the point-weak* topology. Let β be an arbitrary finite subset of $\mathcal{F}$, we set

$$\varphi_\beta : a \in E \to \big(\varphi_{\beta,F}(a)\big)_{F \in \mathcal{F}} \in \prod_{F \in \mathcal{F}} \Phi_F(E),$$

where $\varphi_{\beta,F}(a) = \Phi_F(a)$ if $F \in \beta$ and $\varphi_{\beta,F}(a) = 0$ if $F \notin \beta$. Then we have $\varphi_\beta(E) \subseteq \bigoplus_{F \in \beta} M_{n(F)}$. The set I of all finite subsets β in $\mathcal{F}$ has a natural ordering given by $\beta \preccurlyeq \beta'$ if

$\beta \subseteq \beta'$. If we let $\mathcal{V}$ be an ultrafilter on I with respect to this ordering, we get $\lim_{\mathcal{V}} \varphi_\beta = \Phi$ in the point-weak* topology. Since ψ_α is weak* continuous, then we have

$$\lim_\alpha \lim_{\mathcal{V}} \psi_\alpha \circ \varphi_\beta = \lim_\alpha \psi_\alpha \circ \big(\lim_{\mathcal{V}} \varphi_\beta\big) = \lim_\alpha \psi_\alpha \circ \Phi = \Psi \circ \Phi = id_E$$

in the point-weak* topology.

(3) $\Rightarrow$ (1). For any finite-dimensional subspace E of V, it follows from (3) that there exist complete contractions $\varphi_{i,E} : E \to M_{n(i,E)}$, $\psi_{i,E} : \varphi_{i,E}(E) \to V$ such that $\psi_{i,E} \circ \varphi_{i,E} \to id_E$ in the point-weak* topology. Now for any operator space W and any left weak* continuous contractive linear functional $F : V \check{\otimes} W \to \mathbf{C}$, we want to show that its right weak* continuous extension $F_{\mathrm{r}} : V \otimes W^{**} \to \mathbf{C}$ is contractive with respect to the injective tensor norm. We set

$$F_{i,E} = F \circ (\psi_{i,E} \otimes id_W) : \varphi_{i,E}(E) \check{\otimes} W \to \mathbf{C}.$$

Since $\varphi_{i,E}(E)$ is a finite-dimensional subspace of $M_{n(i,E)}$, $\varphi_{i,E}(E)$ is (weak*) exact and $F_{i,E}$ is a left weak* continuous contraction. It follows from Proposition 3.2 that its right weak* continuous extension

$$(F_{i,E})_{\mathrm{r}} : \varphi_{i,E}(E) \check{\otimes} W^{**} \to \mathbf{C}$$

is contractive. We define

$$G_{i,E} = (F_{i,E})_{\mathrm{r}} \circ (\varphi_{i,E} \otimes id_{W^{**}}) : E \check{\otimes} W^{**} \to \mathbf{C}.$$

Then it is clear that $G_{i,E}$ is contractive and has a norm-preserving extension

$$\widetilde{G}_{i,E} : V \check{\otimes} W^{**} \to \mathbf{C}.$$

By the compactness of $(V \check{\otimes} W^{**})_1^*$, $\widetilde{G}_{i,E}$ has a weak* cluster point $\widetilde{G}_E$ in $(V \check{\otimes} W^{**})_1^*$. For any $v \in E$ and $w^{**} \in W^{**}$, there exists a net $\{w_\alpha\} \subseteq W$ such that $w_\alpha \to w^{**}$ in the $\sigma(W^{**}, W^*)$ topology. Thus we have

$$\begin{aligned}
\widetilde{G}_E(v \otimes w^{**}) &= \lim_i \widetilde{G}_{i,E}(v \otimes w^{**}) \\
&= \lim_i (F_{i,E})_{\mathrm{r}} \circ (\varphi_{i,E} \otimes id_{W^{**}})(v \otimes w^{**}) \\
&= \lim_i (F_{i,E})_{\mathrm{r}}\big(\varphi_{i,E}(v) \otimes w^{**}\big) \\
&= \lim_i \lim_\alpha (F_{i,E})_{\mathrm{r}}\big(\varphi_{i,E}(v) \otimes w_\alpha\big) \\
&= \lim_i \lim_\alpha F_{i,E}\big(\varphi_{i,E}(v) \otimes w_\alpha\big) \\
&= \lim_i \lim_\alpha F \circ (\psi_{i,E} \otimes id_W) \circ (\varphi_{i,E} \otimes id_W)(v \otimes w_\alpha) \\
&= \lim_i \lim_\alpha F\big(\psi_{i,E} \circ \varphi_{i,E}(v) \otimes w_\alpha\big) \\
&= \lim_i F_{\mathrm{r}}\big(\psi_{i,E} \circ \varphi_{i,E}(v) \otimes w^{**}\big) \\
&= F_{\mathrm{r}}(v \otimes w^{**}),
\end{aligned}$$

where all limits converge with respect to the corresponding weak* topologies, the fourth equalities and the eighth equalities follow from the bi-weak* continuity of $(F_{i,E})_{\mathrm{r}}$ and F_{r}, and the last equation follows from $\psi_{i,E} \circ \varphi_{i,E} \to id_E$ in the point-weak* topology. Similarly, there exists a point-weak* cluster G of the net $\{\widetilde{G}_E\} \subseteq (V \check{\otimes} W^{**})^*_1$. From the above proof, we know that for any finite-dimensional subspace $E \subseteq V$,

$$\widetilde{G}_E|_{E \otimes W^{**}} = F_{\mathrm{r}}|_{E \otimes W^{**}}.$$

This implies that

$$G|_{V \otimes_\vee W^{**}} = F_{\mathrm{r}}|_{V \otimes_\vee W^{**}}.$$

Since $G \in (V \check{\otimes} W^{**})^*_1$, F_{r} is contractive with respect to the injective tensor norm. Then it follows from the Proposition 3.2 that V satisfies weak* condition C$'$. □

Theorem 3.3 shows that a dual operator space V satisfies weak* condition C$'$ if and only if it satisfies a very nice local weak* exactness condition. Therefore, we also say that V is *weak* exact* in this case. If $V = M$ is a von Neumann algebra, it is easy to see from Ozawa's proof in [16, Theorem 2] that condition (3) in our Theorem 3.3 is equivalent to his condition (2) in [16, Theorem 2]. Therefore, we may easily obtain the following result.

Proposition 3.4. *Let M be a von Neumann algebra. Then M satisfies weak* condition* C$'$ (*i.e. M is weak* exact*) *if and only if M is weakly exact.*

Let $F : V \otimes W \to \mathbf{C}$ be a linear functional, which is bounded in each component. We can similarly define a unique left weak* continuous linear functional $F_{\mathrm{l}} : V^{**} \otimes W \to \mathbf{C}$ given by

$$\langle F_{\mathrm{l}}, v^{**} \otimes w\rangle = \big\langle v^{**}, F(\cdot \otimes w)\big\rangle \tag{3.3}$$

for all $v^{**} \in V^{**}$ and $w \in W$. This induces a canonical inclusion

$$\tau_{\mathrm{l}} : V^{**} \otimes W \to (V \check{\otimes} W)^{**}$$

and we can obtain the *left augmented injective tensor product* $V^{**} :\!\check{\otimes}\, W$. An operator space V is said to satisfy *condition* C$''$ (which is equivalent to the *local reflexivity* of V) if the identity map on $V^{**} \otimes W$ extends to a completely isometric isomorphism

$$V^{**} :\!\check{\otimes}\, W = V^{**} \check{\otimes} W$$

for all W. We also note that if $F \in (V \check{\otimes} W)^*$, the corresponding operator $\varphi_F : V \to W^*$ is weakly compact (in fact, it can be factored through a Hilbert space) and thus two bi-weak* continuous linear functionals $(F_{\mathrm{r}})_{\mathrm{l}}$ and $(F_{\mathrm{l}})_{\mathrm{r}}$ coincide on $V^{**} \otimes W^{**}$ (see [2, Section 1.9] and Han [9]). In this case, we can obtain a canonical inclusion

$$\tau : V^{**} \otimes W^{**} \to (V \check{\otimes} W)^{**}$$

given by

$$\big\langle \tau(v^{**} \otimes w^{**}), F\big\rangle = \big\langle (F_{\mathrm{l}})_{\mathrm{r}}, v^{**} \otimes w^{**}\big\rangle = \big\langle (F_{\mathrm{r}})_{\mathrm{l}}, v^{**} \otimes w^{**}\big\rangle.$$

Therefore, we can obtain the *augmented injective tensor product* $V^{**} :\check{\otimes}: W^{**}$ and we say that V has *condition* C if we have the complete isometry

$$V^{**} :\check{\otimes}: W^{**} = V^{**} \check{\otimes} W^{**}$$

for all operator spaces W.

Lemma 3.5. *Suppose that V is a locally reflexive operator space. Then for any operator space W, we have the complete isometry*

$$V^{**} \overset{\sigma,\vee}{\otimes}: W^{**} = V^{**} :\check{\otimes}: W^{**}.$$

Proof. Let us first note that V is locally reflexive if and only if for any operator space W, we have the complete isometries

$$(V \check{\otimes} W)^* = \mathcal{I}(W, V^*) = B^\sigma(V^{**} \check{\otimes} W, \mathbf{C}),$$

where the first equality follows from the local reflexivity of V and the second equality follows from Theorem 2.1. The isomorphism $(V \check{\otimes} W)^* = B^\sigma(V^{**} \check{\otimes} W, \mathbf{C})$ is given by

$$F \in (V \check{\otimes} W)^* \to F_{\mathrm{l}} \in B^\sigma(V^{**} \check{\otimes} W, \mathbf{C}).$$

Therefore, we obtain

$$\langle \tau(v^{**} \otimes w^{**}), F \rangle = \langle (F_{\mathrm{l}})_{\mathrm{r}}, v^{**} \otimes w^{**} \rangle = \langle F_{\mathrm{l}}(v^{**} \otimes \cdot), w^{**} \rangle$$

for all $v^{**} \in V^{**}$, $w^{**} \in W^{**}$ and $F \in (V \check{\otimes} W)^*$. This shows that the inclusion

$$\tau : V^{**} \otimes W^{**} \hookrightarrow (V \check{\otimes} W)^{**}$$

coincides with

$$\tau_{\mathrm{r}}^\sigma : V^{**} \otimes W^{**} \hookrightarrow B^\sigma(V^{**} \check{\otimes} W, \mathbf{C})^* = (V \check{\otimes} W)^{**}$$

on $V^{**} \otimes W^{**}$. Therefore, the local reflexivity of V implies the complete isometry $V^{**} \overset{\sigma,\vee}{\otimes}: W^{**} = V^{**} :\check{\otimes}: W^{**}$. $\square$

Kirchberg showed in [14] that a C*-algebra B is exact if and only if B is locally reflexive and B^{**} is weakly exact. The following result shows that the corresponding result holds for operator spaces.

Theorem 3.6. *An operator space V is exact if and only if V is locally reflexive and V^{**} is weak* exact (i.e. V^{**} satisfies weak* condition* C′).

Proof. Suppose that the operator space V is exact. Then it is known from [7] that V is locally reflexive and satisfies condition C. Thus we get the complete isometries

$$V^{**} \check{\otimes} W^{**} = V^{**} :\check{\otimes}: W^{**} = V^{**} \overset{\sigma,\vee}{\otimes}: W^{**}$$

by Lemma 3.5. This shows that V^{**} satisfies weak* condition C′.

Conversely, suppose that V is locally reflexive and V^{**} satisfies weak* condition C′. For any operator space W, it follows from Lemma 3.5 that we have the complete isometries

$$V^{**} \check{\otimes} W^{**} = V^{**} \overset{\sigma,\vee}{\otimes}: W^{**} = V^{**} :\check{\otimes}: W^{**}.$$

So V satisfies the condition C. Therefore, V is exact. □

4. Weak* exact W*-TROs

In the recent development of operator space theory, there is an increasing interest in the study of ternary rings of operators. A ternary ring of operators (or simply, TRO) between Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ is a norm closed subspace V of $\mathcal{B}(\mathcal{K},\mathcal{H})$, which is closed under the triple product

$$(x, y, z) \in V \times V \times V \to xy^*z \in V.$$

A TRO $V \subseteq \mathcal{B}(\mathcal{K},\mathcal{H})$ is called a W^*-TRO if it is weak* closed (equivalently, weak operator closed, or strong operator closed) in $\mathcal{B}(\mathcal{K},\mathcal{H})$. TROs were first introduced by Hestenes [11], and have been intensively studied by Harris [10], Zettl [23], Effros, Ozawa, Ruan [7], Kaur, Ruan [12] and Ruan [19]. Given a TRO $V \subseteq \mathcal{B}(\mathcal{K},\mathcal{H})$, we let $V^\sharp = \{x^* \in \mathcal{B}(\mathcal{H},\mathcal{K}): x \in V\}$ denote the *adjoint space* of V. Then $V^\sharp$ is again a TRO. We let $VV^\sharp$ and $V^\sharp V$ denote the linear spans of vw^* and v^*w for all $v, w \in V$, respectively. Then $VV^\sharp$ and $V^\sharp V$ are $*$-subalgebras of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, and we let

$$C(V) = \overline{VV^\sharp}^{\|\cdot\|} \quad \text{and} \quad D(V) = \overline{V^\sharp V}^{\|\cdot\|}$$

denote the C*-algebras generated by $VV^\sharp$ and $V^\sharp V$, respectively.

If V is a TRO contained in $\mathcal{B}(\mathcal{K},\mathcal{H})$, then

$$A(V) = \begin{bmatrix} C(V) & V \\ V^\sharp & D(V) \end{bmatrix}$$

is the C*-subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ generated by V via the canonical TRO-inclusion

$$\iota_V : v \in V \to \iota_V(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}).$$

It is known from [8,12] that $A(V)$ is uniquely determined by V (up to TRO-isomorphisms) and is just the C*-*envelope* of V. We call $A(V)$ the *linking* C*-*algebra* of V.

If we let $M(V)$ and $N(V)$ denote the multiplier C*-algebras of $C(V)$ and $D(V)$, respectively, we may identify V with the off-diagonal corner of the unital C*-algebra

$$R(V)=\begin{bmatrix} M(V) & V \\ V^\sharp & N(V) \end{bmatrix}.$$

If V is a W^*-TRO contained in $\mathcal{B}(\mathcal{K},\mathcal{H})$, then it is known from [7,12] that $R(V)=A(V)''$ is a von Neumann algebra. In this case, we call $R(V)$ the *linking von Neumann algebra* of V. If we let

$$e=\begin{bmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e^\perp=\begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathcal{K}} \end{bmatrix}$$

denote the corresponding projections on $\mathcal{H}$ and $\mathcal{K}$, respectively, then we may identify V with the off-diagonal corner $\iota_V(V)$ of $R(V)$ and we can write

$$V=eR(V)e^\perp.$$

We can also identify von Neumann algebras $M(V)$ and $N(V)$ with $eR(V)e$ and $e^\perp R(V)e^\perp$ and identify $V^\sharp$ with $e^\perp R(V)e$.

Without loss of generality, we may always assume that a TRO is non-degenerately represented on Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, i.e. $V\mathcal{K}$ is norm dense in $\mathcal{H}$ and $V^\sharp\mathcal{H}$ is norm dense in $\mathcal{K}$. More details about TROs can be found in [7,12,19].

Now if V is a W*-TRO, we let $\iota_V : V \hookrightarrow R(V)$ be the canonical inclusion. Then $V \check{\otimes} B^{**}$ can be identified with the off-diagonal corner, i.e. (1, 2)-corner, $(e\otimes 1)(R(V)\check{\otimes} B^{**})(e^\perp\otimes 1)$ in $R(V)\check{\otimes} B^{**}$ via the inclusion map $\iota_V\otimes id_{B^{**}}$ and $M(V)\check{\otimes} B^{**}$ can be identified with the upper diagonal corner, i.e. (1, 1)-corner, $(e\otimes 1)(R(V)\check{\otimes} B^{**})(e\otimes 1)$ in $R(V)\check{\otimes} B^{**}$. Therefore, the multiplication in $R(V)\check{\otimes} B^{**}$ determines a completely contractive left $M(V)\check{\otimes} B^{**}$ module structure on $V\check{\otimes} B^{**}$. Given $u\in M(V)\otimes B^{**}$, and $w\in V\check{\otimes} B^{**}$ we get $uw\in V\check{\otimes} B^{**}$ such that

$$\sup\bigl\{\|uw\|_{V\check{\otimes}B^{**}}\colon \|w\|_{V\check{\otimes}B^{**}}\leqslant 1\bigr\}\leqslant \|u\|_{M(V)\check{\otimes}B^{**}}.$$

This is also true if we take $w=[w_\alpha]\in M_{1,I}(V\check{\otimes} B^{**})$ as a $1\times I$ row contraction.

To prove the reverse inequality, we need the following fact for W^*-TROs. It is known from the proof of [19, Theorem 3.1] that there exists a net of partial isometries $\{v_\alpha\}_{\alpha\in I}$ in V such that $v_\alpha v_\alpha^*$ are mutually orthogonal projections in $M(V)$ with $\sum_{\alpha\in I} v_\alpha v_\alpha^* = 1_{M(V)}$. Then $[w_\alpha]=[v_\alpha\otimes 1]$ is a row contraction in $M_{1,I}(V\check{\otimes} B^{**})$ such that

$$\begin{aligned}\bigl\|u\bigl[(v_\alpha\otimes 1)\bigr]\bigr\|_{M_{1,I}(V\check{\otimes}B^{**})} &= \left\|u\biggl(\sum_{\alpha\in I} v_\alpha v_\alpha^*\otimes 1\biggr)u^*\right\|^{\frac{1}{2}}_{M(V)\check{\otimes}B^{**}} \\ &= \|uu^*\|^{\frac{1}{2}}_{M(V)\check{\otimes}B^{**}} = \|u\|_{M(V)\check{\otimes}B^{**}}.\end{aligned}$$

This shows that

$$\|u\|_{M(V)\check{\otimes}B^{**}}=\sup\bigl\{\bigl\|[uw_\alpha]\bigr\|_{M_{1,I}(V\check{\otimes}B^{**})}\colon \bigl\|[w_\alpha]\bigr\|_{M_{1,I}(V\check{\otimes}B^{**})}\leqslant 1\bigr\}. \tag{4.1}$$

We can similarly prove that

$$\|u\|_{M(V)\overset{\sigma,\vee}{\otimes}B^{**}} = \sup\left\{\left\|[uw_\alpha]\right\|_{M_{1,I}(V\overset{\sigma,\vee}{\otimes}B^{**})} : \left\|[w_\alpha]\right\|_{M_{1,I}(V\overset{\sigma,\vee}{\otimes}B^{**})} \leqslant 1\right\}. \tag{4.2}$$

To see this, let us first remark that since we can identify

$$V \check{\otimes} B = (e\otimes 1)\big(R(V)\check{\otimes} B\big)\big(e^{\perp}\otimes 1\big),$$

we obtain

$$B^{\sigma}(V\check{\otimes} B, \mathbf{C}) = \big(e^{\perp}\otimes 1\big)\cdot B^{\sigma}\big(R(V)\check{\otimes} B, \mathbf{C}\big)\cdot (e\otimes 1)$$

and thus

$$B^{\sigma}(V\check{\otimes} B, \mathbf{C})^* = (e\otimes 1)\cdot B^{\sigma}\big(R(V)\check{\otimes} B, \mathbf{C}\big)^*\cdot\big(e^{\perp}\otimes 1\big).$$

This shows that

$$V\overset{\sigma,\vee}{\otimes} B^{**} = (e\otimes 1)\big(R(V)\overset{\sigma,\vee}{\otimes} B^{**}\big)\big(e^{\perp}\otimes 1\big).$$

Therefore, $V\overset{\sigma,\vee}{\otimes} B^{**}$ is a TRO, which can be identified with the $(1,2)$-corner in the unital C*-algebra $R(V)\overset{\sigma,\vee}{\otimes} B^{**}$ (see Proposition 2.6). Similarly, $M(V)\overset{\sigma,\vee}{\otimes} B^{**}$ is a unital C*-subalgebra, which can be identified with the $(1,1)$-corner in $R(V)\overset{\sigma,\vee}{\otimes} B^{**}$. We also note that if $\{v_\alpha\}$ is a net of partial isometries in V as we discussed above, then $\{v_\alpha\otimes 1\}$ is a net of partial isometries in $V\overset{\sigma,\vee}{\otimes} B^{**}$ such that

$$\sum_{\alpha\in I}(v_\alpha\otimes 1)(v_\alpha\otimes 1)^* = \sum_{\alpha\in I}\big(v_\alpha v_\alpha^*\otimes 1\big) = 1\otimes 1$$

in $M(V)\overset{\sigma,\vee}{\otimes} B^{**}$. Then we can get (4.2) as we discussed for (4.1).

Kaur and the second author proved in [12, Theorem 4.4] that a TRO V is exact if and only if its linking C*-algebra $A(V)$ is exact. The following is a weak* analogue for W*-TROs.

Theorem 4.1. *Let V be a W*-TRO.*

(1) *If V is weak* exact, then $M(V)$ and $N(V)$ are also weak* exact.*
(2) *V is weak* exact if and only if its linking von Neumann algebra $R(V)$ is weak* exact.*

Proof. We first note from Proposition 2.3 that if M is a von Neumann algebra, then we have the completely isometric injection

$$M\overset{\sigma,\vee}{\otimes} W^{**} \to M\overset{\sigma,\vee}{\otimes}\mathcal{B}(\mathcal{H})^{**}$$

for any operator space $W \subseteq \mathcal{B}(\mathcal{H})$. By the injectivity of the injective operator space tensor product $\check{\otimes}$, we can conclude that M is weak* exact, i.e. M satisfies weak* condition C′, if and only if we have the (completely) isometric isomorphism

$$M \overset{\sigma,\vee}{\otimes}: B^{**} = M \check{\otimes} B^{**}$$

for all unital C*-algebras B.

Now let us assume that V is weak* exact. For any unital C*-algebra B, we have the complete isometry

$$V \check{\otimes} B^{**} = V \overset{\sigma,\vee}{\otimes}: B^{**}.$$

Therefore, we can conclude from (4.1) and (4.2) that

$$\|u\|_{M(V)\check{\otimes}B^{**}} = \|u\|_{M(V)\overset{\sigma,\vee}{\otimes}: B^{**}}$$

for any $u \in M(V) \otimes B^{**}$. This shows that

$$M(V) \check{\otimes} B^{**} = M(V) \overset{\sigma,\vee}{\otimes}: B^{**}.$$

Therefore, $M(V)$ satisfies weak* condition C′ and thus is weak* exact. We can similarly prove that for any unital C*-algebra B,

$$N(V) \check{\otimes} B^{**} = N(V) \overset{\sigma,\vee}{\otimes}: B^{**}.$$

Therefore, $N(V)$ is also weak* exact. This proves (1).

To prove (2), we note that $[M(V), V] = \{[u, v]\colon\ u \in M(V),\ v \in V\}$ is a W*-TRO in $R(V)$ such that

$$N\big(\big[M(V), V\big]\big) = R(V).$$

Since

$$\begin{aligned}
\big[M(V), V\big] \check{\otimes} B^{**} &= \big[M(V) \check{\otimes} B^{**}, V \check{\otimes} B^{**}\big] \\
&= \big[M(V) \overset{\sigma,\vee}{\otimes}: B^{**}, V \overset{\sigma,\vee}{\otimes}: B^{**}\big] \\
&= \big[M(V), V\big] \overset{\sigma,\vee}{\otimes}: B^{**},
\end{aligned}$$

we can conclude from the above discussion that

$$R(V) \check{\otimes} B^{**} = N\big(\big[M(V), V\big]\big) \check{\otimes} B^{**} = N\big(\big[M(V), V\big]\big) \overset{\sigma,\vee}{\otimes}: B^{**} = R(V) \overset{\sigma,\vee}{\otimes}: B^{**}.$$

Therefore, $R(V)$ is weak* exact. The converse implication is an immediate consequence of Proposition 3.1. $\square$

As pointed out by Kirchberg [14] (also see Ozawa [16]), a von Neumann algebra M is weak exact if it contains a weak* dense exact C*-subalgebra A. The corresponding result is also true for W*-TROs.

Corollary 4.2. *If V is a W*-TRO containing a weak* dense exact TRO V_0, then V is weak* exact.*

Proof. Let us first note that the weak* closure, weak operator closure and strong operator closure are all the same for non-degenerate TROs (see [12,23]). If V_0 is an exact TRO, then its linking C*-algebra $A(V_0)$ is exact by [12, Theorem 4.4] and $A(V_0)$ is weak* dense in $R(V)$. This implies that $R(V)$ is weak* exact and thus V is weak* exact. □

5. The structure of exact W*-TROs

It is known from Kirchberg [14] (respectively, from Wassermann [21]) that a von Neumann algebra M is exact (respectively, nuclear) if and only if M is subhomogeneous, i.e. we can write

$$M = \prod_{k=1}^{m} \big(M_{n_k} \mathbin{\overline{\otimes}} L_\infty(X_k, \mu_k)\big),$$

where m and n_k are positive integers and $L_\infty(X_k, \mu_k)$ are abelian von Neumann algebras. The goal of this section is to study the structure of exact (respectively, nuclear) W*-TROs. Without loss of generality, we assume that all W*-TROs considered in this section have a separable predual. The corresponding results for general (not necessarily separable) W*-TROs can be obtained analogously.

Let us first recall from [19] that a W*-TRO V is said to be *stable* if it is TRO-isomorphic to $\mathcal{B}(\ell_2(\mathbf{N})) \mathbin{\overline{\otimes}} V$ and every stable W*-TRO is TRO-isomorphic to its linking von Neumann algebra. Therefore, stable W*-TROs cannot be exact. This class of W*-TROs includes those of type $\mathrm{I}_{\infty,\infty}$, type $\mathrm{II}_{\infty,\infty}$ and type III (see [19, Corollary 4.3]). Moreover, it was shown in [19] that as in the von Neumann algebra case, every W*-TRO V can be uniquely decomposed into the direct sum

$$V = V_{\mathrm{I}} \oplus_\infty V_{\mathrm{II}} \oplus_\infty V_{\mathrm{III}}$$

of W*-TROs of type I, type II, and type III. A W*-TRO V is said to be of type I, type II or type III if its linking von Neumann algebra $R(V)$ is of type I, type II, or type III. Moreover, we may consider W*-TROs of type $\mathrm{I}_{n,m}$ with cardinal numbers n and m, and consider W*-TROs of type $\mathrm{II}_{\lambda,\mu}$ with $\lambda, \mu = 1$ or ∞.

Proposition 5.1. *W*-TROs of type* II *are not exact.*

Proof. We have seen from the above discussion that a W*-TRO of type $\mathrm{II}_{\infty,\infty}$ is not exact. Now if V is a W*-TRO of type $\mathrm{II}_{1,\infty}$ (respectively, type $\mathrm{II}_{\infty,1}$), then there exists an index set I such that V is TRO-isomorphic to the row space $M_{1,I}(M(V))$ (respectively, V is TRO isomorphic to the column space $M_{I,1}(N(V))$ of the von Neumann algebra $M(V)$ (respectively, $N(V)$) associated with V (see [19, Theorem 4.4]). Since exactness is a local property (i.e. if V is exact, then any subspace of V must be exact), V cannot be exact.

Finally, we show that W*-TROs of type $\mathrm{II}_{1,1}$ are not exact. Indeed, since V is a nice normal Hilbert module, there exists a partial isometry $0 \neq v \in V$ such that $e = vv^* \neq 0$ is a projection in $M(V)$ and $f = v^*v \neq 0$ is a projection in $N(V)$. The following map

$$T : exf \in eVf \to exfv^* = exv^*e \in eM(V)e$$

is a TRO-isomorphism from the sub-TRO eVf of V onto the von Neumann subalgebra $eM(V)e$ of $M(V)$, which has a TRO-inverse

$$S : eye \in eM(V)e \to eyev = eyvf \in eVf.$$

This shows that V contains a sub-TRO, which is TRO-isomorphic to a type II_1 von Neumann algebra. Therefore, V cannot be exact. □

The only possible exact W*-TROs must be non-stable and of type I. In this case, we can write (up to TRO-isomorphism)

$$V = \prod_{k=1}^{\infty} (M_{n_k,m_k} \overline{\otimes} L_\infty(X_k, \mu_k)), \tag{5.1}$$

where for each $k \in \mathbf{N}$, n_k and m_k are positive integers or ∞, but not both equal to ∞ at the same time. A W*-TRO is said to be *row subhomogeneous* (respectively, *column subhomogeneous*) if it is of type I with $\sup_k\{n_k\} < \infty$ (respectively, $\sup_k\{m_k\} < \infty$).

Lemma 5.2. *Every row subhomogeneous* (*respectively, column subhomogeneous*) *W*-TRO is nuclear.*

Proof. Let us consider the case when V is a row subhomogeneous W*-TRO, i.e. we have

$$V = \prod_{k=1}^{\infty} \bigl(M_{n_k,m_k} \overline{\otimes} L_\infty(X_k, \mu_k)\bigr)$$

with $1 \leqslant m_k \leqslant \infty$, but $n = \sup_k n_k < \infty$. The column subhomogeneous case can be proved similarly. Let us first assume that

$$V = \prod_{k=1}^{\infty} M_{1,m_k}.$$

For each $k \in \mathbf{N}$, we let $P_k : V \to M_{1,m_k}$ denote the completely contractive projection from V onto the row Hilbert space M_{1,m_k}. If L is an arbitrary finite-dimensional subspace of V with $l = \dim L < \infty$, then $P_k(L)$ is a finite-dimensional subspace of M_{1,m_k}. It is clear that $P_k(L)$ is again a row Hilbert space with dimension $\dim P_k(L) \leqslant l$. Therefore, we may (completely isometrically) identify $P_k(L)$ with an operator subspace of the l-dimensional row Hilbert space R_l. So we can obtain the following completely isometric embeddings:

$$L \to \prod_{k=1}^{\infty} P_k(L) \to \prod_{k=1}^{\infty} R_l = R_l \overline{\otimes} \ell_\infty(\mathbf{N}).$$

Since $R_l \overline{\otimes} \ell_\infty(\mathbf{N}) = R_l \check{\otimes} \ell_\infty(\mathbf{N})$ is a nuclear operator space, this shows that L is an exact operator space. Therefore, V is exact. Moreover, it is obvious that V is an injective W*-TRO and thus has the WEP (weak expectation property). Since an operator space is nuclear if and only if it is exact (or locally reflexive) and has the WEP (see [7, Theorem 4.5]), we can conclude that V is a nuclear operator space.

It follows from the above discussion that

$$V = M_{n,\infty} \overline{\otimes} \ell_\infty(\mathbf{N}) = \prod_{k=1}^{\infty} M_{n,\infty} = \prod_{k=1}^{\infty} (M_{n,1} \check{\otimes} M_{1,\infty}) = M_{n,1} \check{\otimes} \left(\prod_{k=1}^{\infty} M_{1,\infty} \right)$$

is nuclear. If $L_\infty(X, \mu)$ is a commutative von Neumann algebra with a separable predual, it is known from the Hahn–Banach theorem that we may (completely) isometrically identify $L_\infty(X, \mu)$ with a (completely) contractively complemented (operator) subspace of $\ell_\infty(\mathbf{N})$. Therefore, $M_{n,\infty} \overline{\otimes} L_\infty(X, \mu)$ is nuclear since it can be completely isometrically identified with a completely contractively complemented operator subspace of $M_{n,\infty} \overline{\otimes} \ell_\infty(\mathbf{N})$. The same idea can be applied to prove that

$$V = \prod_{k=1}^{\infty} \big(M_{n_k, m_k} \overline{\otimes} L_\infty(X_k, \mu_k) \big) \subseteq \prod_{k=1}^{\infty} \big(M_{n,\infty} \overline{\otimes} \ell_\infty(\mathbf{N}) \big) = M_{n,\infty} \overline{\otimes} \ell_\infty(\mathbf{N} \times \mathbf{N})$$

is a nuclear operator space. $\square$

Now we are ready to state our main theorem in this section.

Theorem 5.3. *Let V be a W*-TRO. Then the following are equivalent*:

(1) *V is nuclear*;
(2) *V is exact*;
(3) *we can decompose V into the direct sum*

$$V = V_{\mathrm{r}} \oplus_\infty V_{\mathrm{c}}$$

of a row subhomogeneous W-TRO V_{r} and a column subhomogeneous W*-TRO V_{c}.*

Proof. (1) $\Rightarrow$ (2) is obviously and (3) $\Rightarrow$ (1) follows from Lemma 5.2. We only need to prove (2) $\Rightarrow$ (3).

Assume that V is an exact W*-TRO. Then we must have

$$V = \prod_{k=1}^{\infty} \big(M_{n_k, m_k} \overline{\otimes} L_\infty(X_k, \mu_k) \big),$$

where for each k, n_k and m_k are not both equal to ∞. We first claim that there are no subsequences $n_{k_j} \to \infty$ and $m_{k_j} \to \infty$ simultaneously. Suppose that there exist $n_{k_j} \to \infty$ and $m_{k_j} \to \infty$ simultaneously. Then we can find a copy of type I von Neumann algebra $M = M_1 \oplus M_2 \oplus \cdots$ contained in V and thus V is not be exact, which is a contradiction.

Next, we claim that there exists a (disjoint) partition $I \cup J = \mathbf{N}$ of $\mathbf{N}$ such that $\sup\{n_{k_i} \mid i \in I\} < \infty$ and $\sup\{m_{k_j} \mid j \in J\} < \infty$. Indeed, assume that this is not true. We let

$I^l = \{i \in \mathbf{N}\colon\ n_{k_i} \leqslant l\}$ for each $l \in \mathbf{N}$. Then there exist infinitely many positive integers, say, $l_1 < l_2 < \cdots$, such that

$$\sup\bigl\{m_{k_j}\colon\ j \in \bigl(I^{l_i}\bigr)^c = \mathbf{N} \setminus I^{l_i}\bigr\} = \infty.$$

Then for each $i \in \mathbf{N}$, there exists $j_i \in (I^{l_i})^c$ such that $m_{k_{j_i}} > l_i$. In this case, we must have $n_{k_{j_i}} > l_i$. Then we can apply an induction procedure to pick up sequences $n_{k_{j_i}}, m_{k_{j_i}}$ with

$$n_{k_{j_1}} < n_{k_{j_2}} < \cdots \to \infty \quad \text{and} \quad m_{k_{j_1}} < m_{k_{j_2}} < \cdots \to \infty.$$

This is a contradiction and the assertion is proved.

Let $I \cup J = \mathbf{N}$ be a (disjoint) partition of $\mathbf{N}$ discussed in the above claim. Then $V_{\mathrm{r}} = \prod_{i \in I}(M_{n_{k_i}, m_{k_i}} \overline{\otimes} L_\infty(X_{k_i}, \mu_{k_i}))$ is a row subhomogeneous W*-TRO and $V_{\mathrm{c}} = \prod_{j \in J}(M_{n_{k_j}, m_{k_j}} \overline{\otimes} L_\infty(X_{k_j}, \mu_{k_j}))$ is a column subhomogeneous W*-TRO. In this case, we can write $V = V_{\mathrm{r}} \oplus_\infty V_{\mathrm{c}}$. □

Since every nuclear dual operator space is injective and thus is a W*-TRO, we can obtain the following corollary.

Corollary 5.4. *Let V be a dual operator space. Then V is nuclear if and only if V is an exact* (*respectively, nuclear*) *W*-TRO.*

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